Student seminar exercise sheet Week 10

Let K/F be a Galois extension of number fields and write $G = \mathrm{Gal}(K/F)$.

1. Looking at the proof of Lemma 4.1 in the book of Childress, construct the function δ_1, prove it is well-defined and prove that $\mathrm{Im}(\delta_1) = \ker(f_0)$.

2. If A, B are G-modules such that their Herbrand quotient exist, show that

$$\mathcal{Q}_G(A \times B) = \mathcal{Q}_G(A)\mathcal{Q}_G(B).$$

3. Let K/F be an Galois extension of number fields with Galois group G.

   (a) Let $v \in V_F$ be a place of F. Prove that G acts transitively on the set $\{w \mid v\}$.

   (b) Check that the action of G on J_K coincides with the usual action of G on $K^\times$.

   (c) Let $v \in V_F$, $w|v$ be fixed. Show that the map

$$\begin{array}{ccc} \mathrm{Gal}(K_w/F_v) & \longrightarrow & G \\ \sigma & \longmapsto & \sigma_{|K} \end{array}$$

   is an isomorphism onto the subgroup $G_w = \{\sigma \in G \mid \sigma w = w\}$ of G.

4. Recall Hilbert's Theorem 90 from class: For any Galois extension of field K/F with Galois group G and $f : G \to K^\times$ satisfying $f(\tau\sigma) = \tau(f(\sigma))f(\tau)$, there is some $\alpha \in K^\times$ such that $f(\sigma) = \sigma(\alpha)/\alpha$ for all $\sigma \in G$. Use this to show

   (a) There is an exact sequence of abelian groups

$$1 \to F^\times \to K^\times \to \mathrm{KerNm}_{K/F} \to 1.$$

   (b) Deduce the the norm map is surjective for an extension of finite fields.